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OPTIMUM AUTOFRETTAGE PRESSURE IN THICK CYLINDERS

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ABSTRACT

In optimal design of pressurised thick-walled cylinders, an increase in the allowable internal pressure can be achieved by an autofrettage process. An analysis is carried out on plain cylinders by using the von Mises and Tresca yield criteria to develop a procedure in which the autofrettage pressure is determined analytically, resulting in a reduced stress distribution. A validation by a numerical simulation shows that the analytical and numerical simulations correlate well in terms of trend and magnitude of stresses.

Keywords: *Autofrettage, pressure vessels, residual stress, plastic collapse, finite element analysis*

1.0 INTRODUCTION

Autofrettage is a common process of producing residual stresses in the wall of a usually thick-walled cylinder prior to use. An appropriate pressure, large enough to cause yielding within the wall, is applied to the inner wall of the cylinder and then removed. Large scale yielding occurs in the autofrettaged thick-walled cylinder wall, [1]. Upon the release of this pressure, a compressive residual circumferential stress is developed to a certain radial depth at the bore. These residual stresses serve to reduce the tensile stresses developed as a result of subsequent application of an operating pressure, thus increasing the load bearing capacity, [2], [3].

Due to the ever-increasing industrial demand for axisymmetric pressure vessels which have wide applications in chemical, nuclear, fluid transmitting plants, power plants and military equipment, the attention of designers has been concentrated on this particular branch of engineering. The increasing scarcity and high cost of materials have led researchers not to confine themselves to the customary elastic regime but attracted their attention to the elastic-plastic approach which offers more efficient use of materials, [4].

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2.0 CYLINDER SUBJECTED TO INTERNAL PRESSURE

For a cylinder subjected to an internal pressure, P_i, the radial stress, σ_r, and circumferential stress, σ_θ, distributions are given by Lame's formulation:

$$\sigma_\theta = \frac{P_i}{k^2 - 1}\left[1 - \frac{r_o^2}{r^2}\right] \tag{1}$$

and

$$\sigma_\theta = \frac{P_i}{k^2 - 1}\left[1 + \frac{r_o^2}{r^2}\right] \tag{2}$$

For a cylinder with end caps and free to change in length, the axial stress is given by [5]:

$$\sigma_z = \frac{P_i}{k^2 - 1} \tag{3}$$

3.0 YIELD CRITERIA

According to the Tresca yield theory, yielding occurs when the Tresca equivalent stress is [5]:

$$\sigma_{Tr} = (\sigma_\theta - \sigma_r) = \sigma_Y \tag{4}$$

Based on the von Mises yield theory, yielding occurs when the von Mises equivalent stress is [6]:

$$\sigma_{vM} = \sqrt{\frac{1}{2}\left\{(\sigma_\theta - \sigma_r)^2 + (\sigma_r - \sigma_z)^2 + (\sigma_z - \sigma_\theta)^2\right\}} = \sigma_Y \tag{5}$$

Two important pressure limits, $P_{Y,i}$ and $P_{Y,o}$, are considered to be of importance in the study of pressurised cylinders. $P_{Y,i}$ corresponds to the internal pressure required at the onset of yielding at the inner surface of the cylinder, and $P_{Y,o}$ is the internal pressure required to cause the wall thickness of cylinder to yield completely. The magnitudes of $P_{Y,i}$ and $P_{Y,o}$, according to Tresca yield criterion are, [2], [6], [7]:

$$P_{Tr,Y,i} = \frac{(k^2 - 1)}{2k^2}\sigma_Y \tag{6}$$

$$P_{Tr,Y,o} = \frac{(k^2 - 1)}{2}\sigma_Y \tag{7}$$

and based on the von Mises yield criterion, the magnitudes of $P_{Y,\,i}$ and $P_{Y,\,o}$ are, [2]:

$$P_{vM,Y,i} = \frac{(k^2-1)}{\sqrt{3}k^2}\sigma_Y \tag{8}$$

$$P_{vM,Y,o} = \frac{(k^2-1)}{\sqrt{3}}\sigma_Y \tag{9}$$

Equations (4) and (5) give the relation between the von Mises and Tresca equivalent stresses for the state of stress in a pressurised thick-walled cylinder:

$$\sigma_{vM} = \frac{\sqrt{3}}{2}\left(\sigma_\theta - \sigma_r\right) = \frac{\sqrt{3}}{2}\sigma_{Tr} \tag{10}$$

and shows that the Tresca criterion is more conservative than the von Mises criterion by 15.5 %.

4.0 RESIDUAL STRESSES

If the internal pressure is removed after part of the cylinder thickness has become plastic, a residual stress is set up in the wall. Assuming that during unloading the material follows Hooke's Law, the residual stresses can be obtained from the equations below. For the plastic region, $r_i \leq r \leq r_a$, the respective residual stresses in the radial, hoop and axial directions are [8]:

$$\sigma_{r,p,R} = \frac{\sigma_Y}{2}\left\{\left[2\ln\left(\frac{r}{r_a}\right)-1+\frac{m^2}{k^2}\right]-\left[2\ln\,(m+1)-\frac{m^2}{k^2}\right]\left(\frac{1}{k^2-1}\right)\left(1-\frac{r_o^2}{r^2}\right)\right\} \tag{11a}$$

$$\sigma_{\theta,p,R} = \frac{\sigma_Y}{2}\left\{\left[2+2\ln\left(\frac{r}{r_a}\right)-1+\frac{m^2}{k^2}\right]-\left[2\ln\,(m+1)-\frac{m^2}{k^2}\right]\left(\frac{1}{k^2-1}\right)\left(1+\frac{r_o^2}{r^2}\right)\right\} \tag{11b}$$

$$\sigma_{z,p,R} = \frac{\sigma_Y}{2}\left\{\left[1+2\ln\left(\frac{r}{r_a}\right)-1+\frac{m^2}{k^2}\right]-\left[2\ln\,(m+1)-\frac{m^2}{k^2}\right]\left(\frac{1}{k^2-1}\right)\right\} \tag{11c}$$

For the elastic region, $r_a \leq r \leq r_o$, the respective residual stresses in the radial, hoop and axial directions are:

$$\sigma_{r,e,R} = \frac{\sigma_Y}{2}\left[1-\frac{r_o^2}{r^2}\right]\left\{\frac{m^2}{k^2}-\left(1-\frac{m^2}{k^2}+2\ln\,(m)\right)\left(\frac{1}{k^2-1}\right)\right\} \tag{12a}$$

$$\sigma_{\theta,e,R} = \frac{\sigma_Y}{2}\left[1+\frac{r_o^2}{r^2}\right]\left\{\frac{m^2}{k^2}-\left(1-\frac{m_2}{k^2}+2\ln(m)\right)\left(\frac{1}{k^2-1}\right)\right\} \tag{12b}$$

$$\sigma_{z,e,R} = \frac{\sigma_Y}{2}\left\{\frac{m^2}{k^2}-\left(1-\frac{m^2}{k^2}+2\ln(m)\right)\left(\frac{1}{k^2-1}\right)\right\} \tag{12c}$$

where $m=\frac{r_a}{r_i}$ and r_a is the autofrettage radius. By substituting $r = r_a$ in Equations (12a-c), the residual stresses at the junction radius r_a are obtained:

$$\sigma_{r,R} = \frac{\sigma_Y}{2}\left[1-\frac{k^2}{m^2}\right]\left\{\frac{m^2}{k^2}-\left(1-\frac{m^2}{k^2}+2\ln(m)\right)\left(\frac{1}{k^2-1}\right)\right\} \tag{13a}$$

$$\sigma_{\theta,R} = \frac{\sigma_Y}{2}\left[1+\frac{k^2}{m^2}\right]\left\{\frac{m^2}{k^2}-\left(1-\frac{m^2}{k^2}+2\ln(m)\right)\left(\frac{1}{k^2-1}\right)\right\} \tag{13b}$$

$$\sigma_{z,R} = \frac{\sigma_Y}{2}\left\{\frac{m^2}{k^2}-\left[1-\frac{m^2}{k^2}+2\ln(m)\right]\left(\frac{1}{k^2-1}\right)\right\} \tag{13c}$$

The residual stress distributions are shown in Figures 1 and 2.

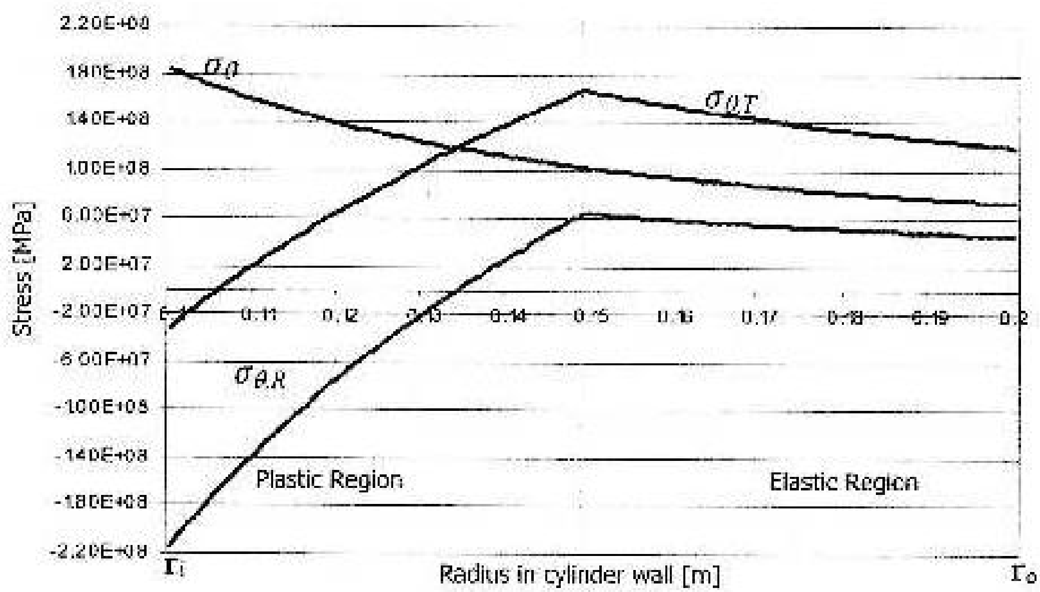

Figure 1: Hoop stresses (σ_θ) due to operating pressure, residual autofrettage pressure ($\sigma_{\theta,R}$) and total ($\sigma_{\theta,T}$)

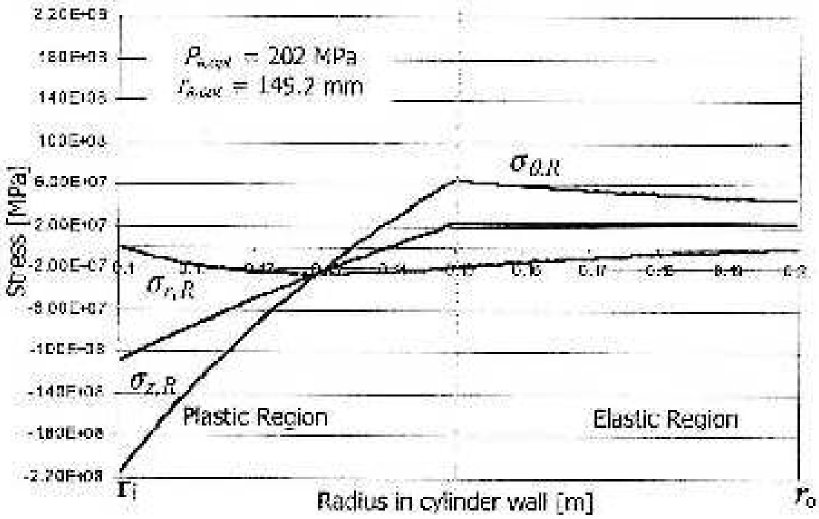

Figure 2: Residual stress distributions in cylinder wall

On application of the operating pressure the total stress of the partially autofrettaged cylinder is the summation of the residual stress and the stress due to the operating pressure, i.e.:

$$\sigma_{r,T} = \sigma_{r,R} - \sigma_{r,opr} \tag{14a}$$

$$\sigma_{\theta,T} = \sigma_{\theta,R} + \sigma_{\theta,opr} \tag{14b}$$

$$\sigma_{z,T} = \sigma_{z,R} + \sigma_{z,opr} \tag{14c}$$

The above total stresses are shown in Figure 3.

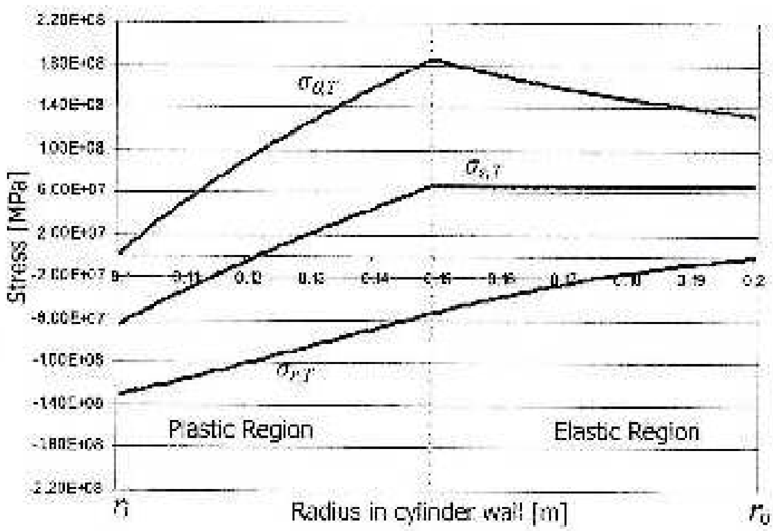

Figure 3: Total hoop, radial and axial stress distributions in cylinder wall when subjected to operating pressure, after autofrettage

Hence at $r = r_a$, when the cylinder is subjected to an internal operating pressure, after being treated by autofrettage, the Tresca equivalent stress at the elastic-plastic junction is:

$$\sigma_{Tr} = \sigma_Y \frac{k^2}{m^2}\left[\frac{m^2}{k^2} - (1 - \frac{m^2}{k^2} + 2\ln(m))\frac{1}{k^2-1}\right] + \left[\frac{2P_{opr}}{k^2-1}\right]\left(\frac{k^2}{m^2}\right) \quad (15)$$

Differentiating σ_{Tr} with respect to m and equating the differential to zero:

$$\frac{d\sigma_{Tr}}{dm} = \sigma_Y\left[\frac{-2k^2}{m^3} + \frac{2k^2}{m^3} - \frac{4k^2}{m^3}\ln(m)\right] - 4P_{opr}\frac{k^2}{m^3} = 0$$

to obtain

$$m = \exp(\frac{P_{opr}}{\sigma_Y}) \quad (16)$$

Since $\frac{d^2\sigma_{Tr}}{dr_a^2} \rangle 0$, the r_a ($= r_{a,opt}$) obtained is the optimum and minimum autofrettage radius. Letting $n = \frac{P_{opr}}{\sigma_Y}$,

therefore,

$$m_{Tr} = \exp(n) \qquad \text{- Tresca} \quad (17)$$

and

$$m_{vM} = \exp\left(\frac{\sqrt{3}}{2}n\right) \qquad \text{- von Mises} \quad (18)$$

Figure 4 shows how the optimum autofrettage radius is influenced by the operating pressure. It shows that for a certain operating pressure, the optimum autofrettage radius obtained using Tresca yield criteria is more than that using von Mises criterion.

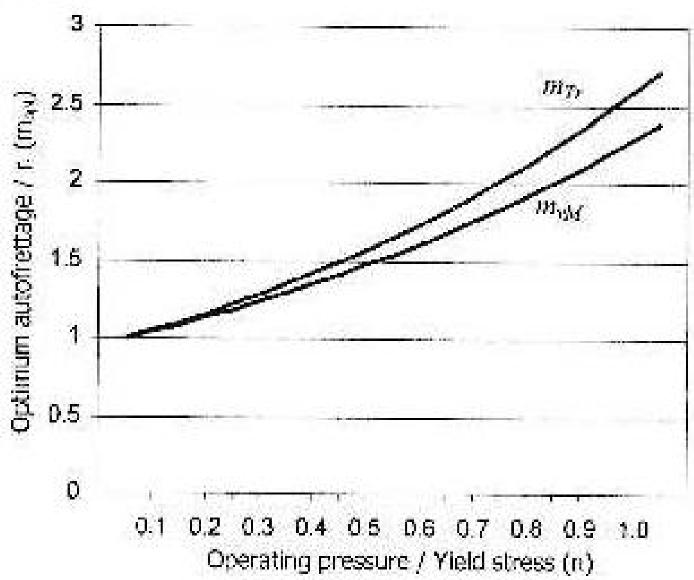

Figure 4: Effect of yield criteria on optimum autofrettage radius

5.0 MAXIMUM INTERNAL PRESSURE OF AUTOFRETTAGED CYLINDER

Equations (6) and (7) are used to obtain the (Tresca) autofrettage pressure to cause different stages of yielding in a virgin cylinder. For a cylinder treated with partial autofrettage, the internal pressure to cause the inner surface to yield again can be obtained. Substituting Equations (1), (2) and (12) into Equation (14), and using Tresca yield criterion, when $r = r_i$, the internal pressure to cause yielding at the inner surface is,

$$P_{Y,i} = \frac{\sigma_Y}{2}\left[2\ln(m) + 1 - \frac{m^2}{k^2}\right] \tag{19}$$

and when $r = r_o$, by substituting Equations (1), (2) and (11) into Equation (14), and using Tresca yield criterion, the internal pressure to cause the whole wall thickness to yield is,

$$P_{Y,o} = \frac{\sigma_Y}{2}\left[2\ln(m) + k^2 - m^2\right] \tag{20}$$

Figures 5 and 6 respectively show the maximum internal pressure to cause yielding at the inner surface and to cause the whole cylinder to yield. These pressures are influenced by different optimum autofrettage pressure levels which were obtained when an operating pressure was initially known. The internal pressure to cause yielding at the inner surface of a cylinder which is treated with optimum autofrettage pressure, is greater than that for a non-treated cylinder (Figure 5). On the other hand, the internal pressure to cause full yielding in a cylinder which has been treated with optimum autofrettage, is lower than that which is non-autofrettaged (Figure 6).

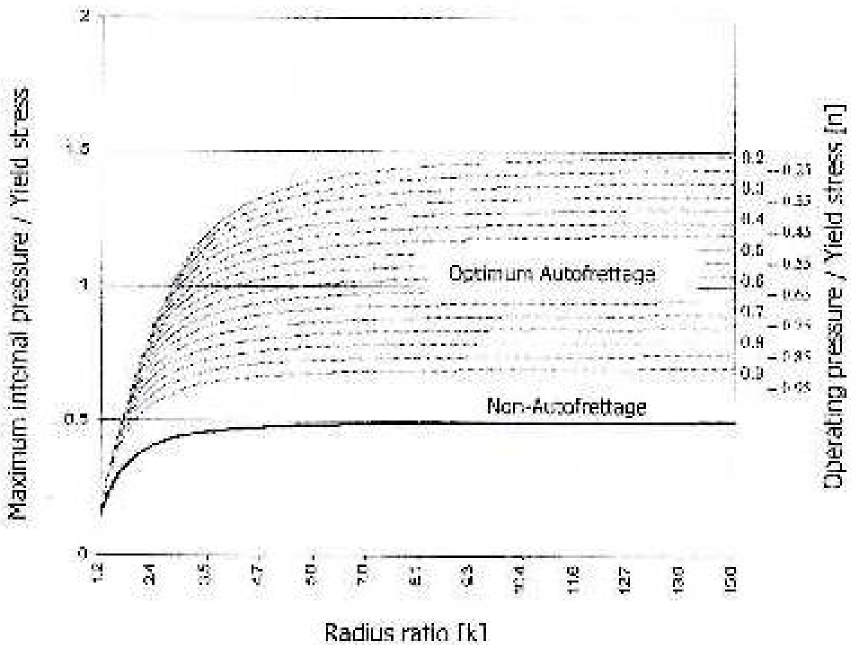

Figure 5: Maximum internal pressure to cause inner surface to yield, with different optimum autofrettage levels- Tresca

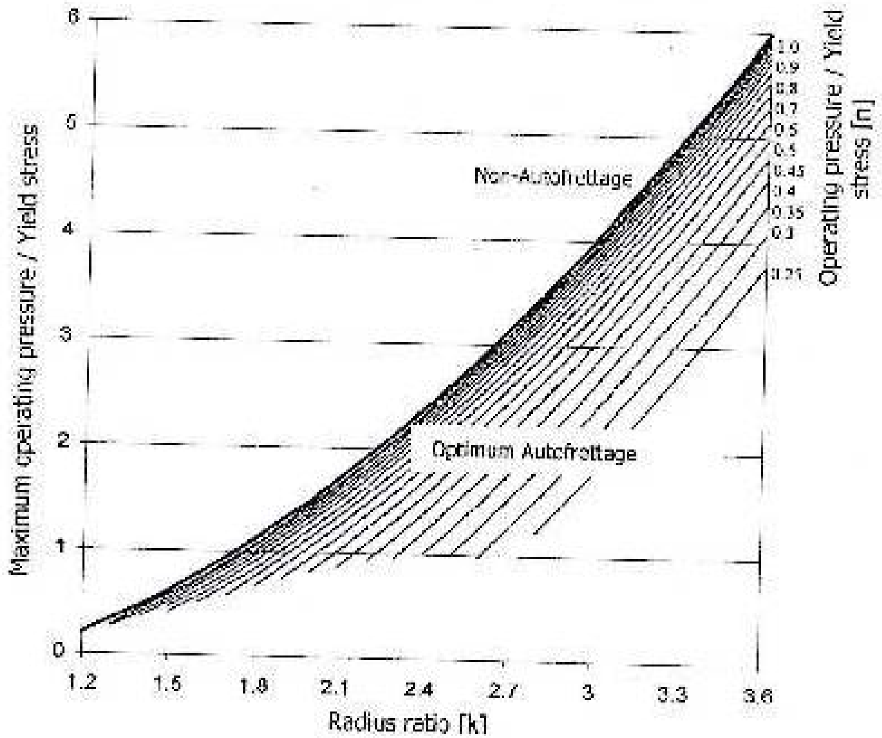

Figure 6: Maximum internal pressure to cause the whole wall to yield, with different optimum autofrettage levels-Tresca

6.0 FULLY AUTOFRETTAGED CYLINDER

A special case is when the cylinder is fully autofrettaged, i.e. $r_a = r_o$. Therefore $m = k$ and the Tresca equivalent stress at any radius can be obtained from Equation (15):

$$\sigma_{Tr} = \sigma_Y \left[1 - \frac{2\ln(k)}{k^2 - 1} \left(\frac{r_o^2}{r^2} \right) \right] + \frac{2P_{opr}}{k^2 - 1} \left(\frac{r_o^2}{r^2} \right) \tag{21}$$

Therefore the internal pressure to cause the internal surface and whole wall to yield is obtained by substituting $r = r_i$, $r = r_o$ and $m = k$ in Equations (19) and (20). The comparison of allowable internal pressures of a cylinder treated with full and non-autofrettage, are shown in Table 1 and in Figures 7 and 8. Figure 7 shows that full autofrettage is beneficial if yielding of the inner surface is required, in which case the cylinder can sustain the highest internal pressure. To cause the whole wall to yield, the cylinder should not be autofrettaged, in which case the cylinder can sustain the highest internal pressure, as shown in Figure 8.

Table 1: Comparison between allowable internal pressures on non-autofrettaged, optimumly autofrettaged and fully autofrettaged thick-walled cylinder

Autofrettage Level	Internal pressure to cause the inner surface to yield	P_i/σ_Y	Internal pressure to cause the whole cylinder wall to yield	P_i/σ_Y
Non autofrettage	$\frac{(k^2-1)}{2k^2}\sigma_Y$	0.375	$\frac{(k^2-1)}{2}\sigma_Y$	1.5
Optimum autofrettage	$\frac{\sigma_Y}{2}\left[2\ln(m)+1-\frac{m^2}{k^2}\right]$	0.622	$\frac{\sigma_Y}{2}\left[2\ln(m)+k^2-m^2\right]$	1.287
Full autofrettage	$\sigma_Y \ln k$	0.693	$\sigma_Y \ln k$	0.693

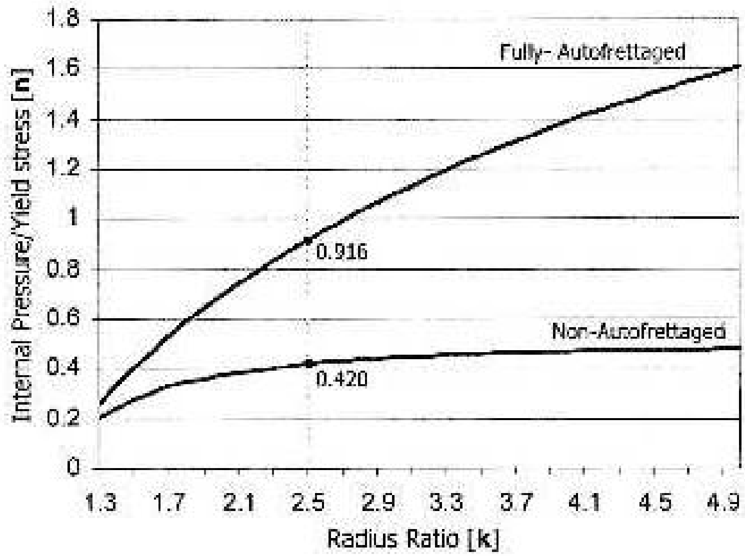

Figure 7: Maximum internal pressure to cause internal surface to yield

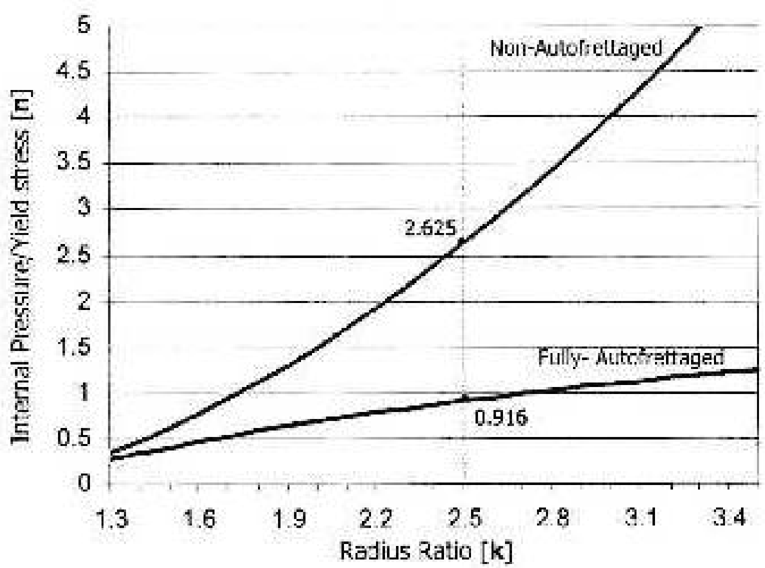

Figure 8: Maximum internal pressure to cause whole thickness to yield

7.0 THEORETICAL OPTIMUM AUTOFRETTAGE PRESSURE

The autofrettage pressure P_a is a sufficiently high internal pressure applied before a cylinder is put into use by applying an operating pressure. The radius of the elastic-plastic junction line is called the autofrettage radius r_a. The objective is to design for a total minimum equivalent stress at the junction line. The value of autofrettage pressure which satisfies this condition is called the *Optimum Autofrettage Pressure,* $P_{a,\,opt}$ and the radius of elastic-plastic junction line is called the *Optimum Autofrettage Radius,* $r_{a,\,opt}$.

The internal pressure to cause (Tresca) yielding to a depth of r is:

$$P=\frac{\sigma_Y}{2}\left[1-\frac{r^2}{r_o^2}+2\ln\left(\frac{r}{r_i}\right)\right] \tag{22}$$

From Equations (17) and (18) the optimum autofrettage radius is deduced as,

$$r_{a,opt}=r_i e^{n} \qquad \text{- Tresca}$$

$$r_{a,opt}=r_i e^{\frac{\sqrt{3}}{2}n} \qquad \text{- von Mises}$$

Therefore the optimum autofrettage pressure is:

$$P_{a,opt,Tr}=\frac{\sigma_Y}{2}\left[1-\frac{e^{2n}}{k^2}+2n\right] \qquad \text{- Tresca} \tag{23}$$

$$P_{a,opt,vM}=\frac{\sigma_Y}{2}\left[1-\frac{e^{\sqrt{3}n}}{k^2}+\sqrt{3}n\right] \qquad \text{- von Mises} \tag{24}$$

The above optimum autofrettage pressures result in the minimum equivalent stress and occurs on the elastic-plastic junction line as shown in Figure 9.

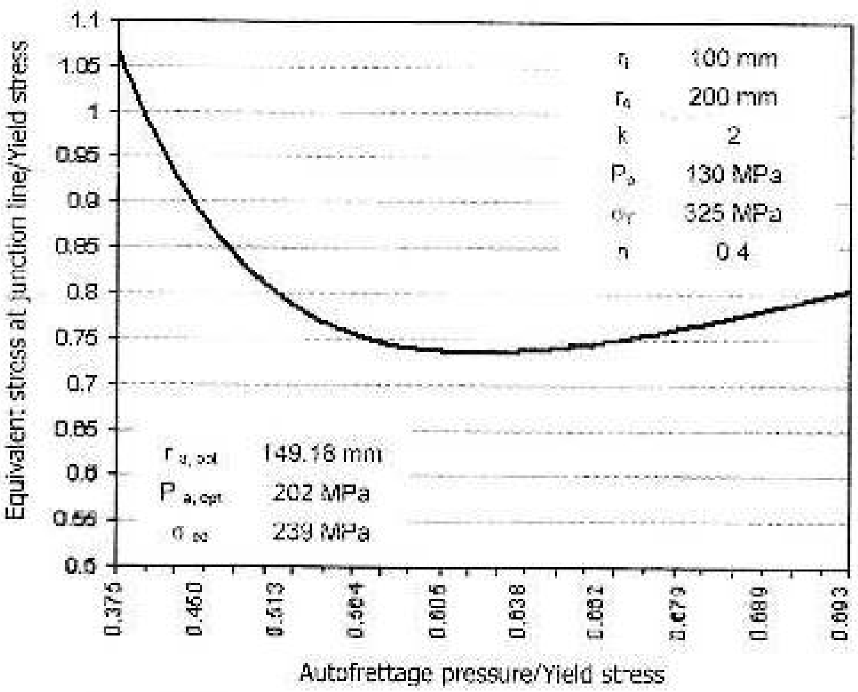

Figure 9: Optimum autofrettage pressure and radius

8.0 OPTIMUM AUTOFRETTAGE AND MAXIMUM OPERATING PRESSURES

The relation between the optimum autofrettage pressure and operating pressure of thick-walled pressurised cylinders can be obtained:

$$\frac{P_{a,opt,Tr}}{P_{opr,Tr}} = \left[1 + \frac{k^2 - e^{2n}}{2nk^2}\right] \qquad \text{- Tresca} \qquad (25)$$

Figure 10 shows the optimum autofrettage pressure/operating pressure ratio varying with the radius ratio, using Tresca yield criterion. For thick walled cylinders, increasing the operating pressure leads to an increase in the optimum autofrettage pressure, which in turn leads to an increase in autofrettage radius.

$$\frac{P_{a,opt,vM}}{P_{opr,vM}} = \left[\frac{\sqrt{3}}{2} + \frac{k^2 - e^{\sqrt{3}n}}{nk^2}\right] \qquad \text{- von Mises} \qquad (26)$$

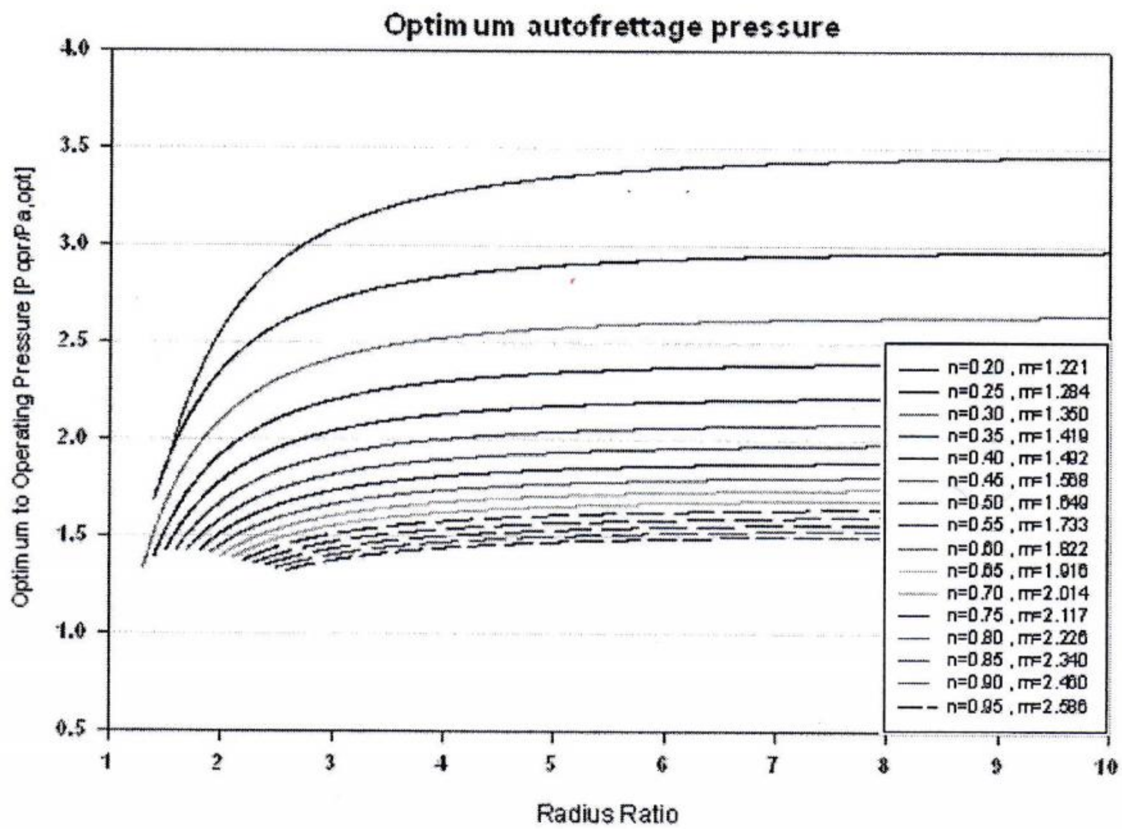

Figure10: Optimum autofrettage for different values of operating pressure and radius ration

9.0 FINITE ELEMENT ANALYSIS

The autofrettage process may be simulated by Finite Element Method, making use of elastic-plastic analysis. It is possible to model the autofrettage process by applying pressure to the inner surface of the model, removing it and then calculating the residual stress field, followed by reloading with an operating pressure.

Using a 2D axisymmetric element available in ABAQUS v6.5 [9] a finite element mesh of a cylinder with an inside radius 100 mm and outside radius of 200 mm was generated. An autofrettage pressure of 202 MPa was applied, and then removed. The residual stress distributions were evaluated in the thick-walled cylinder. The operating pressure of 130 MPa was then applied. The von Mises equivalent stress was used in the subsequent analysis. The material used was steel and this material has the following properties:

$$E = 203 \text{ GPa}$$
$$\sigma_Y = 325 \text{ MPa}$$
$$\nu = 0.33$$

The material is assumed to be isotropic, linearly elastic and has bilinear kinematic hardening using von Mises plasticity response.

9.1 FEM Results

Using the above FE cylinder model and comparing between Tresca and von Mises criteria, the autofrettaged thick-walled cylinder give the following results on the difference in the optimum autofrettage radius and pressure.

	Tresca criterion	von Mises criterion
m	1.492	1.414
$r_{as\,opt}$	149.20 mm	141.42 mm
$P_{as\,opt}$	202 MN/m^2	194 MN/m^2

Table 1 shows the influence of autofrettage level on the allowable internal pressure, using Tresca yield criterion, the different levels being cylinders not treated with autofrettage, treatment with optimum autofrettage and autofrettaged until the whole cylinder has yielded.

10.0 CONCLUSIONS

The following conclusions are thus drawn:

1) The autofrettage process increases the maximum allowable internal pressure.
2) The autofrettage process cannot increase the maximum internal pressure to cause the whole thickness of the cylinder to yield.
3) If the operating pressure, P_{ope}, is large, the optimum boundary radius ($r_{ii,\,opt}$) is also large.

4) If the yield stress, σ_Y, is large, the optimum boundary radius ($r_{a,opt}$) is small.
5) The optimum autofrettage pressure causes the lowest equivalent stress during application of operating pressure, and this occurs at the elastic-plastic junction line.
6) The optimum autofrettage radius, $r_{a,opt}$ depends on the operating pressure P_{opr}, and the inner radius of the thick-wall cylinder r_i, apart from the material property σ_Y.

NOMENCLATURE

P	pressure
r	radius
t	thickness
k	outer:inner radius ratio
m	autofrettage:inner radius ratio
n	operating pressure:yield stress ratio
σ	normal stress
τ	shear stress

Subscripts

i	inner
o	outer
a	autofrettage
r	radial
θ	hoop
z	axial
Y	yield
p	plastic
e	elastic
opt	optimum
opr	operating
max	maximum
min	minimum
Tr	Tresca
vM	von Mises
R	residual
T	total

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